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Superradiative scattering magnons*

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Abstract. A magnon-photon interaction for the magnetic vector of the electromagnetic wave perpendicular to the direction of magnetization in a ferromagnet is constructed. The magnon part of the interaction is reduced with the use of Bogoliubov transformation. The resulting magnon-photon interaction is found to contain several interesting new radiation effects. The self energy of the magnon is calculated and life times arising from the radiation scattering are predicted. The magnon frequency shift due to the radiation field is found. One of the terms arising from the one-magnon one-photon scattering gives a line width in reasonable agreement with the experimentally measured value of ferromagnetic reasonance line width in yettrium iron garnet. Surface magnon scattering is indicated and the contribution of this type of scattering to the radiative line width is discussed The problem of magnetic superradiance is indicated and it is shown that in anisotropic ferromagnets the emission is proportional to the square of the number of magnons and the divergence is considerably minimized. Accordingly the magnetic superradiance emerges as a hyperradiance with much more radiation intensity than in the case of disordered atomic superradiance.

1. Introduction

Fundamental quantum mechanical treatments of the effect of magnon-photon interaction have been given recently by the present author (Shrivastava 1975, 1976). However, these studies were limited to the uniform $k = 0$ magnon only. In the present paper we give a general treatment valid for finite wave vectors and make further effort to improve and generalize the previous theory. In this effort certain new effects have been found The present study of the interaction for the magnetic vector of the radiation field perpendicular to the direction of magnetization also complements the study of the parallel configuration reported in the preceeding papers (Shrivastava 1978a, b; 1979a-e, 1980).

We find that there are several new radiation effects which occur in lower orders than the so-called Suhl instability effects (Suhl 1956a, 1957b). Our finding is that the Suhl's first-order instability as it is called by Keffer (1966) occurs in the fourth-order perturbation theory whereas "Suhl's second-order instability" can occur only in the sixth-order perturbation theory. These effects

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of Suhl are also reviewed by Haas and Gallen (1963) again with incorrect nomenclature. There are indeed several new effects which occur in the second- or higher-order perturbation theory. While we discuss the new effects in some detail, brief remarks are also made about Suhl's work. We show that atleast for the uniform magnon the contribution of the surface scattering of magnons gives only a very small interference to the radiative effects. We discuss the problem of superradiance from a magnetically ordered material and thereby come to the conclusion of an interesting possibility of a hyperradiance which is more dense than the Dicke superradiance from disordered systems. We consider that the magnetic vector of the electromagnetic wave is perpendicular to the direction of magnetization in a ferromagnetic material. In this case the interaction of interest can be written as $g\mu_B h_x S_x$. We use the expanded version of the Holstein-Primakoff representation (Oguchi 1960) of spin waves in terms of magnetic site variables according to which,

$$S_l^+ = (2S)^{1/2}a_l - \frac{1}{2(2S)^{1/2}}\, a_l^\dagger a_l a_l$$

$$S_l^- = (2S)^{1/2}a_l^\dagger - \frac{1}{2(2S)^{1/2}}\, a_l^\dagger a_l^\dagger a_l \tag{1}$$

Using the relations,

$$S_{xl} = \tfrac{1}{2}(S_l^+ + S_l^-) \tag{2}$$

$$h = i\sum_{q} (2\pi\hbar\omega_q/L^3)^{1/2}(b_q e^{iq.r_l} - b_q^\dagger e^{-iq.r_l}) \tag{3}$$

for the x-component of the spin and the magnetic vector of the radiation field obtained from the second quantization of the Maxwell equations in terms of the creation and annihilation operators of the photons, the interaction can be written

$$\begin{aligned}\mathcal{H}' &= \sum_l g\mu_B h_{xl} S_{xl}\\ &= \sum_l g\mu_B (2\pi\hbar\omega_q/L^3)^{1/2}\{(2S)^{1/2}(a_l + a_l^\dagger)(b_q e^{iq.r_l} - b_q^\dagger e^{-iqr_l})\\ &\quad - \frac{1}{2(2S)^{1/2}}(a_l^\dagger a_l a_l + a_l^\dagger a_l^\dagger a_l)(b_q e^{iq.r_l} - b^\dagger_q e^{-iq.r_l})\}\end{aligned} \tag{4}$$

Introducing the Fourier transforms of the site variables as

$$a_j = N^{-1/2}\sum \exp(-ik.r_j)a_k$$

$$a_j^\dagger = N^{-1/2}\sum \exp(ik.r_j)a^\dagger_k \tag{5}$$

the interaction becomes

$$\begin{aligned}\mathscr{H}' = \sum g\mu_B(2SN)^{1/2}[iA_q\{a^\dagger_k b_q\delta(k-q)+a_k b_q\delta(k+q)\}-iA_q^*\{a_k b^\dagger_q\delta(k-q)\\ +a_k^\dagger b^\dagger_k\delta(k+q)\}]-\sum_{qk_1k_2k_3} g\mu_B\frac{1}{2(2SN)^{1/2}}[iA_q\{a^\dagger_{k_1}a_{k_2}a_{k_3}b_q\\ \times\delta(k_1-k_2-k_3-q)+a^\dagger_{k_1}a^\dagger_{k_2}a_{k_3}b_q\delta(k_1+k_2-k_3-q)\}\\ -iA_q^*\{a^\dagger_{k_1}a_{k_2}a_{k_3}b_q^\dagger\delta(q+k_1-k_2-k_3)+a^\dagger_{k_1}a^\dagger_{k_2}a_{k_3}b_q^\dagger\\ \times\delta(k_1+k_2-k_3+q)\}].\end{aligned} \tag{6}$$

It is useful to introduce the Bogoliubov transformation on the magnons,

$$\begin{aligned}a^\dagger_k &= u_k\alpha^\dagger_k+v_k\alpha_k\\ a_k &= u_k^*\alpha_k+v_k^*\alpha^\dagger_k\end{aligned} \tag{7}$$

which absorbs a large part of the potential into the unperturbed Hamiltonian, such that the remaining potential can be treated as a perturbation. The effect of the transformation (7) on (6) after elaborate algebra leads to,

$$\begin{aligned}\mathscr{H}' = \sum_{kq} g\mu_B(2SN)^{1/2}[iA_q\{(u_k+v_k^*)\alpha^\dagger_k b_q\delta(k-q)+(v_k+u_k^*)\alpha_k b_k\delta(k+q)\}\\ -iA_q^*\{(u_q^*+v_k)\alpha_k b_q^\dagger\delta(k-q)+(v_k^*+u_k)\alpha_k^\dagger b_q^\dagger\delta(k+q)\}]\\ -\sum_{qk_1k_2k_3} g\mu_B\frac{1}{2(2SN)^{1/2}}[\{iA_q(u_{k_1}v^*_{k_2}u_{k_5}+u_{k_1}u_{k_2}u^*_{k_3})\\ \times\alpha^\dagger_{k_1}\alpha^\dagger_{k_2}\alpha_{k_3}b_q\delta(k_1+k_2-k_3-q)-iA_q^*(v_{k_1}u^*_{k_2}v^*_{k_3}+v_{k_1}v_{k_2}v^*_{k_3})\\ \times\alpha_{k_1}\alpha_{k_2}\alpha^\dagger_{k_3}b^\dagger_q\delta(k_1+k_2-k_3-q)+iA_q(v_{k_3}v^*_{k_2}v^*_{k_3}+v_{k_1}u_{k_2}v^*_{k_3})\\ \times\alpha_{k_1}\alpha^\dagger_{k_2}\alpha^\dagger_{k_3}b_q\delta(k_1+q-k_2-k_3)-iA_q^*(u_{k_1}u^*_{k_2}u^*_{k_3}+u_{k_1}v_{k_2}u^*_{k_3})\\ \times\alpha^\dagger_{k_1}\alpha_{k_2}\alpha_{k_3}b_q^\dagger\delta(k_1+q-k_2-k_3)+iA_q(u_{k_1}u^*_{k_2}v^*_{k_3}+u_{k_1}v_{k_2}v^*_{k_3})\\ \times\alpha^\dagger_{k_1}\alpha_{k_2}\alpha^\dagger_{k_3}b_q\delta(k_1+k_3-k_2-q)-iA_q^*(v_{k_1}v^*_{k_2}u^*_{k_3}+v_{k_1}u_{k_2}u^*_{k_3})\\ \times\alpha_{k_1}\alpha^\dagger_{k_2}\alpha_{k_3}b_q^\dagger\delta(k_1+k_3-k_2-q)\}+\{iA_q(u_{k_1}v_{k_2}u^*_{k_3}+u_{k_1}u^*_{k_2}u^*_{k_3})\end{aligned}$$

$$\times\alpha^{\dagger}_{k_1}\alpha_{k_2}\alpha_{k_3}b_q\delta(k_1-k_2-k_3-q)-iA_q^*(v_{k_1}u_{k_2}v^*_{k_3}+v_{k_1}v^*_{k_2}v^*_{k_3})$$

$$\times\alpha_{k_1}\alpha^{\dagger}_{k_2}\alpha^{\dagger}_{k_3}b_q^{\dagger}\delta(k_1-k_2-k_3-q)+iA_q(v_{k_1}v^*_{k_2}u^*_{k_3}+v_{k_1}u_{k_2}u^*_{k_3})$$

$$\times\alpha_{k_1}\alpha^{\dagger}_{k_2}\alpha_{k_3}b_q\delta(k_2-k_1-k_3-q)-iA_q^*(u_{k_1}u^*_{k_2}v^*_{k_3}+u_{k_1}v_{k_2}v^*_{k_3})$$

$$\times\alpha^{\dagger}_{k_1}\alpha_{k_2}\alpha^{\dagger}_{k_3}b_q^{\dagger}\delta(k_2-k_1-k_3-q)+iA_q(v_{k_1}u^*_{k_2}v^*_{k_3}+v_{k_1}v_{k_2}v^*_{k_3})$$

$$\times\alpha_{k_1}\alpha_{k_2}\alpha^{\dagger}_{k_3}b_q\delta(k_3-k_1-k_2-q)-iA_q^*(u_{k_1}v^*_{k_2}u^*_{k_3}+u_{k_1}u_{k_2}u^*_{k_3})$$

$$\times\alpha^{\dagger}_{k_1}\alpha^{\dagger}_{k_2}\alpha_{k_3}b_q^{\dagger}\delta(k_3-k_1-k_2-q)\}+iA_q(v_{k_1}u^*_{k_2}u^*_{k_3}+v_{k_2}v_{k_2}u^*_{k_3})$$

$$\times\alpha_{k_1}\alpha_{k_2}\alpha_{k_3}b_q\delta(k_1+k_2+k_3+q)-iA_q^*(u_{k_1}v^*_{k_2}v^*_{k_3}+u_{k_1}u_{k_2}v^*_{k_3})$$

$$\times\alpha^{\dagger}_{k_1}\alpha^{\dagger}_{k_2}\alpha^{\dagger}_{k_3}b_q^{\dagger}\delta(k_1+k_2+k_3+q)]. \tag{8}$$

The unperturbed Hamiltonian which defines the states is given by

$$\mathscr{H}_0=\sum_k \epsilon_k\alpha_k^{\dagger}\alpha_k+\sum_q \hbar\omega_q b_q^{\dagger}b_q. \tag{9}$$

Since u_k and v_k are real, the interaction (8) can be symmetrized as follows,

$$\begin{aligned}\mathscr{H}'=\sum_{kq} g\mu_B(2SN)^{1/2}[&iA_q\{(u_k+v_k)\alpha_k^{\dagger}b_q\delta(k-q)+(v_k+u_k)\alpha_k b_q\delta(k+q)\}\\ &-iA_q^*\{(u_k+v_k)\alpha_k b_q^{\dagger}\delta(k-q)+(v_k+u_k)\alpha_k^{\dagger}b_q^{\dagger}\delta(k+q)\}]\\ &+\sum_{qk_1k_2k_3}[F(k_1k_2k_3q)\alpha^{\dagger}_{k_1}\alpha^{\dagger}_{k_2}\alpha_{k_3}b_q\delta(k_1+k_2-k_3-q)\\ &+F(k_1k_2k_3q)\alpha_{k_1}\alpha^{\dagger}_{k_2}\alpha^{\dagger}_{k_3}b_q\delta(k_1+q-k_2-k_3)+G(k_1k_2k_3q)\\ &\times\alpha^{\dagger}_{k_1}\alpha_{k_2}\alpha^{\dagger}_{k_3}b_q\delta(k_1+k_3-k_2-q)+F(k_1k_2k_3q)\alpha^{\dagger}_{k_1}\alpha_{k_2}\alpha_{k_3}b_q\delta(k_1-k_2-k_3-q)\\ &+G(k_1k_2k_3q)\alpha_{k_1}\alpha^{\dagger}_{k_2}\alpha_{k_3}b_q\delta(k_2-k_1-k_3-q)+F(k_1k_2k_3q)\alpha_{k_1}\alpha_{k_2}\alpha^{\dagger}_{k_3}b_q\\ &\times\delta(k_3-k_1-k_2-q)+G(k_1k_2k_3q)\alpha_{k_1}\alpha_{k_2}\alpha_{k_3}b_q\delta(k_1+k_2+k_3+q)+h.c.]\end{aligned} \tag{10}$$

where the coupling is given by

$$2F(k_1k_2k_3q) = \phi^{(1)}_{k_1k_2k_3q} + \phi^{(2)}_{k_1k\ k_3q} \tag{11}$$

$$2G(k_1k_2k_3q) = \phi^{(3)}_{k_1k_2k_3q} + \phi^{(4)}_{k_1k_2k_3q} \tag{12}$$

$$\phi^{(1)}_{k_2k_1k_3q} = g\mu_B[2(2SN)^{1/2}]^{-1}(u_{k_1}v_{k_2}u_{k_3} + u_{k_1}u_{k_2}u_{k_3})$$

$$\phi^{(2)}_{k_1k_2k_3q} = g\mu_B[2(2SN)^{1/2}]^{-1}(v_{k_1}u_{k_2}v_{k_3} + v_{k_1}v_{k_2}v_{k_3})$$

$$\phi^{(3)}_{k_1k_2k_3q} = g\mu_B[2(2SN)^{1/2}]^{-1}(u_{k_1}u_{k_2}v_{k_3} + u_{k_1}v_{k_2}v_{k_3})$$

$$\phi^{(3)}_{k_1k_2k_3q} = g\mu_B[2(2SN)^{1/2}]^{-1}(v_{k_1}u_{k_2}u_{k_3} + v_{k_1}v_{k_2}u_{k_3}). \tag{13}$$

This completes the definition of the interaction (10). Thus (9) and (10) together constitute the adequate Hamiltonian. Any kind of quantum mechanics can now be done.

3. Perturbation theoretic results

We first deduce the correction of the interaction to the system energy and from that calculate the self energy for one magnon. The imaginary part of the self energy is then separated to find the radiative contribution to the magnon life time. The contribution of the first four terms of (10) is calculated to give,

$$\Sigma_s^{(1)} = \sum_{kq} g^2\mu_B^{\,2}2SNA_qA_q^*(u_k+v_k)^2\{(N_q-n_k)\delta(k-q)/(\epsilon_k-\hbar\omega_q)$$

$$+(N_q+n_k+1)\delta(k+q)/(\epsilon_k+\hbar\omega\ )\} \tag{14}$$

which gives the following contribution to the one-magnon self energy,

$$\Sigma_k^{(1)} = \sum_q 2g^2\mu_B^{\,2}SN|A_q|^2(u_k+v_k)^2\left\{\frac{\delta(k-q)}{\hbar\omega_q-\epsilon_k} + \frac{\delta(k+q)}{\hbar\omega_q+\epsilon_k}\right\} \tag{15}$$

since $\epsilon_{\mathbf{k}} = \epsilon_{-\mathbf{k}}$. This contribution is represented by the diagrams (a) and (b) in Figure 1. The major terms which conserve the number of particles in (10) give the following contribution to the system energy,

$$
\begin{aligned}
\Sigma_s^{(2)} = \sum_{qk_1k_2k_3} \Big[& \widetilde{F}^2(k_1k_2k_3q)\, \frac{(n_{k_1}+1)(n_{k_2}+1)n_{k_3}N_q - n_{k_1}n_{k_2}(n_{k_3}+1)(N_q+1)}{\epsilon_{k_1}+\epsilon_{k_2}-\epsilon_{k_3}-\hbar\omega_q} \\
& \times\delta(k_1+k_2-k_3-q) + \hat{F}^2(k_1k_2k_3q)\, \frac{N_q n_{k_1}(n_{k_2}+1)(n_{k_3}+1) - n_{k_2}n_{k_3}(n_{k_1}+1)(N_q+1)}{\epsilon_{k_2}+\epsilon_{k_3}-\epsilon_{k_1}-\hbar\omega_q} \\
& \times\delta(k_2+k_3-k_1-q) + \widetilde{G}^2(k_1k_2k_3q)\, \frac{(n_{k_1}+1)n_{k_2}(n_{k_3}+1)N_q - n_{k_1}(n_{k_2}+1)n_{k_3}(N_q+1)}{\epsilon_{k_1}-\epsilon_{k_2}+\epsilon_{k_3}-\hbar\omega} \\
& \times\delta(k_1+k_3-k_2-q)\Big]
\end{aligned}
\tag{16}
$$

where

$$
\begin{aligned}
\widetilde{F}(k_1k_2k_3q) &= F(k_1k_2k_3q)+F(k_2k_1k_3q) \\
\hat{F}(k_1k_2k_3q) &= F(k_1k_2k_3q)+F(k_1k_3k_2q) \\
\widetilde{G}(k_1k_2k_3q) &= G(k_1k_2k_3q)+G(k_3k_2k_1q).
\end{aligned}
\tag{16a}
$$

The result (16) gives the following contribution to the single particle self energy,

$$
\begin{aligned}
\Sigma_{k_1}^{(2)} = \sum_{qk_3k_2} \Big[& \widetilde{F}^2(k_1k_2k_3q)\, \frac{n_{k_3}N_q - n_{k_2}n_{k_3} - n_{k_2}N_q - n_{k_2}}{\epsilon_{k_1}+\epsilon_{k_2}-\epsilon_{k_3}-\hbar\omega_q}\, \delta(k_1+k_2-k_3-q) \\
& + \hat{F}^2(k_1k_2k_3q)\, \frac{N_q n_{k_2} + N_q n_{k_3} - n_{k_2}n_{k_3} + N_q}{\epsilon_{k_2}+\epsilon_{k_3}-\epsilon_{k_1}-\hbar\omega}\, \delta(k_2+k_3-k_1-q) \\
& + \widetilde{G}^2(k_1k_2k_3q)\, \frac{n_{k_2}N_q - n_{k_3}n_{k_2} - n_{k_3}N_q - n_{k_3}}{\epsilon_{k_1}-\epsilon_{k_2}+\epsilon_{k_3}-\hbar\omega_q}\, \delta(k_1+k_3-k_2-q)\Big].
\end{aligned}
\tag{17}
$$

These are represented by (c) in Figure 1. Next we consider the terms in which three particles are created and one annihilated and vice versa. These are dis-

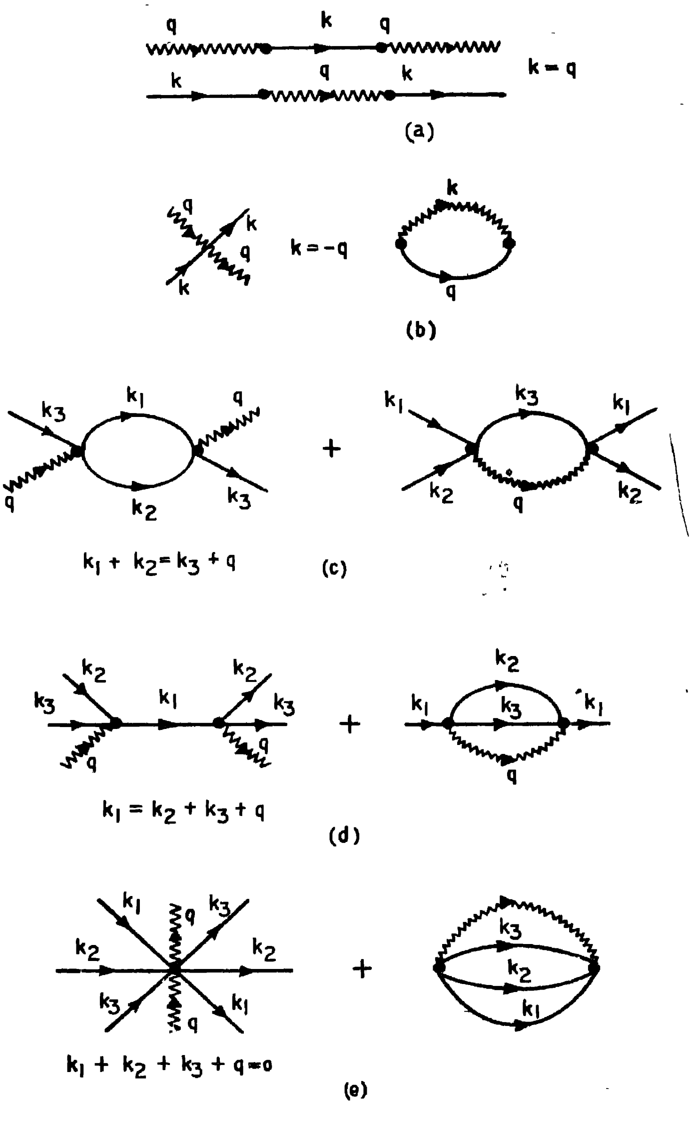

Figure 1. Second-order diagrams originating from the magnon-photon interaction (8). Some of the diagrams obtained by permutation of various wave vectors are not drawn. The wave line inticated a photon and the solid lines a magnon

played by the diagrams (d) in Figure 1, and contribute the following to the system energy,

$$\Sigma_s^{(3)} = \sum_{qk_1k_2k_3} \Big[\hat{F}^2(k_1k_2k_3q)\, \frac{(n_{k_1}+1)n_{k_2}n_{k_3}N_q-(N_q+1)(n_{k_2}+1)(n_{k_3}+1)n_{k_1}}{\epsilon_{k_1}-\epsilon_{k_2}-\epsilon_{k_3}-\hbar\omega_q}$$

$$\times\delta(k_1-k_2-k_3-q)+G^2(k_1k_2k_3q)\, \frac{n_{k_1}(n_{k_2}+1)n_{k_3}N_q-(n_{k_1}+1)n_{k_2}(n_{k_3}+1)(N_q+1)}{-\epsilon_{k_1}+\epsilon_{k_2}-\epsilon_{k_3}-\hbar\omega_q}$$

$$\times\delta(k_2-k_1-k_3-q)+F^2(k_1k_2k_3q)\, \frac{n_{k_1}n_{k_2}(n_{k_3}+1)N_q-(n_{k_1}+1)(n_{k_2}+1)n_{k_3}(N_q+1)}{-\epsilon_{k_1}-\epsilon_{k_2}+\epsilon_{k_3}-\hbar\omega_q}$$

$$\times\delta(k_3-k_1-k_2-q)\Big]. \tag{18}$$

The contribution to the one-magnon self energy from the above is

$$\Sigma_{k_1}^{(3)} = \sum_{qk_2k_3} \Big[\hat{F}^2(k_1k_2k_3q)\, \frac{N_qn_{k_2}+N_qn_{k_3}+n_{k_2}n_{k_3}+N_q+n_{k_2}+n_{k_3}+1}{-\epsilon_{k_1}+\epsilon_{k_2}+\epsilon_{k_3}+\hbar\omega_q}$$

$$\times\delta(k_1-k_2-k_3-q)+\hat{G}^2(k_1k_2k_3q)\, \frac{n_{k_3}N_q-n_{k_2}n_{k_3}-n_{k_2}N_q-n_{k_2}}{-\epsilon_{k_1}+\epsilon_{k_2}-\epsilon_{k_3}-\hbar\omega_q}$$

$$\times\delta(k_2-k_1-k_3-q)+\tilde{F}^2(k_1k_2k_3q)\, \frac{n_{k_2}N_q-n_{k_2}n_{k_3}-n_{k_3}N_q-n_{k_3}}{-\epsilon_{k_1}-\epsilon_{k_2}+\epsilon_{k_3}-\hbar\omega_q}$$

$$\times\delta(k_3-k_1-k_2-q)\Big]. \tag{19}$$

The last term of (10) along with its hermitian conjugate gives rise to only real contribution to the self energy which corresponds to a small contribution to the frequency shift but does not contribute to the magnon life time. The system energy contribution is represented by the diagram (e) in Figure 1. Accordingly the second-order contribution is given by,

$$\Sigma_s^{(4)} = \sum_{qk_1k_2k_3} \hat{G}^2(k_1k_2k_3q)\, \frac{(n_{k_1}+1)(n_{k_2}+1)(n_{k_3}+1)(N_q+1)-n_{k_1}n_{k_2}n_{k_3}N_q}{\epsilon_{k_1}+\epsilon_{k_2}+\epsilon_{k_3}+\hbar\omega_q}$$

$$\times\delta(k_1+k_2+k_3+q). \tag{20}$$

The contribution of which to the self energy is,

$$\Sigma_{k_1}^{(4)} = \sum_{qk_2k_3} \hat{G}^2(k_1k_2k_3q)\, \frac{n_{k_2}n_{k_3}+n_{k_2}N_q+n_{k_3}N_q+n_{k_2}+n_{k_3}+N_q+1}{\epsilon_{k_1}+\epsilon_{k_2}+\epsilon_{k_3}+\hbar\omega_q}$$

$$\times\delta(k_1+k_2+k_3+q). \tag{21}$$

$\hat{G}(k_1k_2k_3q)$ is obtained from $G(k_1k_2k_3q)$ by permuting k_1, k_2, k_3. The total self energy is determined from the sum of (15), (17), (19) and (21). The imaginary part of which gives the magnon life time as,

$$
\begin{aligned}
1/\tau = (2\pi/\hbar)\{[\Sigma\, 2g^2\mu_B{}^2SN\,|\,A_q\,|^2(u_q+v_q)^2\delta(\hbar\omega_0-\epsilon_0)] \\
+ \sum_{qk_2k_3} [\widetilde{F}^2(k_1k_2k_3q)(n_{k_3}N_q-n_{k_2}n_{k_3}-n_{k_2}N_q-n_{k_2})\delta(k_1+k_2-k_3-q) \\
\times\delta(\epsilon_{k_1}+\epsilon_{k_2}-\epsilon_{k_3}-\hbar\omega_q)+\hat{F}^2(k_1k_2k_3q)(N_qn_{k_2}+N_qn_{k_3}-n_{k_2}n_{k_3}+N_q) \\
\times\delta(k_2+k_3-k_1-q)\delta(\epsilon_{k_2}+\epsilon_{k_3}-\epsilon_{k_1}-\hbar\omega_q)+\widetilde{G}^2(k_1k_2k_3q) \\
\times(n_{k_2}N_q-n_{k_3}n_{k_2}-n_{k_3}N_q-n_{k_3})\delta(k_1+k_3-k_2-q)\delta(\epsilon_{k_1}-\epsilon_{k_2}+\epsilon_{k_3}-\hbar\omega_q) \\
+\hat{F}^2(k_1k_2k_3q)(N_qn_{k_2}+N_qn_{k_3}+n_{k_2}n_{k_3}+N_q+n_{k_2}+n_{k_3}+1) \\
\times\delta(k_1-k_2-k_3-q)\delta(-\epsilon_{k_1}+\epsilon_{k_2}+\epsilon_{k_3}+\hbar\omega_q)+\widetilde{G}^2(k_1k_2k_3q) \\
\times(n_{k_3}N_q-n_{k_2}n_{k_3}-n_{k_2}N_q+n_{k_2})\delta(k_2-k_1-k_3-q) \\
\times\delta(-\epsilon_{k_1}+\epsilon_{k_2}-\epsilon_{k_3}-\hbar\omega_q)\widetilde{F}^2(k_1k_2k_3q)(n_{k_2}N_q-n_{k_2}n_{k_3}-n_{k_3}N_q-n_{k_3}) \\
\times\delta(k_3-k_1-k_2-q)\delta(-\epsilon_{k_1}-\epsilon_{k_2}+\epsilon_{k_3}-\hbar\omega_q)]\}
\end{aligned}
\tag{22}
$$

It is seen that there are terms that are independent of temperature and r.f. power. There are others with linear dependence on N_q which determine the r.f. power. There are temperature dependences of $T^{-1/2}$ and $T^{\frac{5}{2}}\,[\exp(h\omega_0/k_B\;T)[-1]$, etc. All the terms have a multiplicative factor of the square of the saturation magnetization and as seen before, they are observable giving rise to important contributions at very low temperatures where normally accepted thermal effects are extinguished. Proper identification requires very low temperatures, of the order of 1 K. There are ten branches in the dispersion relation. The first term of (22) already gives line width which is of correct order of magnitude as already shown by Shrivastava (1979c) and agrees with that measured by Sanders *et al* (1974) for $k = 0$ mode of MnF_2 Unfortunately the measurements of temperature dependence of this width at low temperatures $\simeq$1 K are not available.

We see the evaluation of the first term of (22) for the ferromagnetic resonance in yettrium iron garnet. The factor arising from the transformation is,

$$(u_k+v_k)^2 = 1+\frac{B_k}{\epsilon_k} \tag{23}$$

so the line width is,

$$\Delta H = (2\pi/h)2g^2\mu_B^2 SNA_0^2(1+B_k/\epsilon_k) \tag{24}$$

where B_k is arising from the dipole sum given by Keffer (1966). In most of the system of interest $B_k/\epsilon_k \ll 1$, therefore for $k = 0$, the ferromagnetic resonance for the uniform magnon we have,

$$\Delta H = (4\pi/h)g^2\mu_B^2 SNA_0^2 \simeq 4\pi^2(4\pi M_0)V(\omega/2\pi c)^3. \tag{24a}$$

For yettrium iron garnet, the magnetization is $4\pi M_0 = 1760$ Gauss, so that for a sample diameter of 0·3 cm at $\omega/2\pi = 9$ GHz, a line width of 26·5 Gauss is predicted whereas the experimental value measured by Rezende *et al* (1974) is about 22 Gauss. We have ignored the wave guide factor of the order of unity. As the cube of the frequency is required, small errors in the frequency measurement can lead to comparatively large errors in the width calculation. It therefore appears that the calculated estimate is in reasonable agreement with the experiment. Linear dependence of the line shift on r f power has been measured by Matcovich, Belson and Goldberg (1961) and Matcovich *et al* (1962) and it appears that our prediction is consistent with this measurement.

4. *Surface magnon scattering*

In this section we wish to show that given a simple model which is exact, the contribution of the scattering of magnons at the surface of the sample to the radiation interference is small. By assuming that the spherical surface is full of maximum number of pits possible, Sparks *et al* (1961) obtain the surface potential

$$V_c = \sum_k{}' F_k(a_0^\dagger a_k + a_0 a_k^\dagger) \tag{25}$$

where

$$F_k = \frac{32}{3V}\pi^2 R^3 hN \sum_k (3\cos^2\theta - 1)\frac{j_1(kR)}{kR} \tag{25a}$$

in which R is the radius of a small pit on the surface, j_1 is a spherical Bessel function, and θ_k is the scattering angle If we consider our model Hamiltonian as,

$$\mathcal{H} = \sum \hbar\omega_0 b_0^\dagger b_0 + \sum_k c_k a_k^\dagger a_k + \sum_n i\alpha(b_0^\dagger - b_0)(a_0 + a_0^\dagger) + V_c \tag{26}$$

then we have seen that without any approximation the self energy can be calculated to all orders and diagram expansion is not necessary. In this case the model self energy is (Shrivastava 1975),

$$\Sigma_k = \sum_k \frac{F_k^2}{E-\epsilon_k} - \sum_0 \alpha^2\left(\frac{1}{E+\hbar\omega_0} - \frac{1}{E-\hbar\omega_0}\right) - \sum_0 \alpha^4\left(\frac{1}{E+\hbar\omega_0} - \frac{1}{E-\hbar\omega_0}\right)^2$$

$$\left[E+\epsilon_0 - \sum_k{}' \frac{F_k^2}{E+\epsilon_k} - \sum_0 \alpha^2\left(\frac{1}{E+\hbar\omega_0} - \frac{1}{E-\hbar\omega_0}\right)\right]^{-1} \tag{27}$$

With the help of this result we proceed to see that though there is interference present in (27), it is small. The first term in (27) can be written as,

$$\sum_k{}' \frac{F_k^2}{E-\epsilon_k} = i\Gamma_k + \Delta\epsilon_k$$

so that the line width from the first term is of the order of,

$$-(2/\hbar)\Gamma_k = (2\pi/\hbar)\int_0^{k\,max} |F_k|^2 \rho_k dk$$

evaluated at resonance $E = \epsilon_k$. This can give a width of about $2G$ for a polished sample to about $50G$ for a rough sample. The line shift $\Delta\epsilon$ is of the same order of magnitude. The second term of (27) is real giving

$$-\Delta\epsilon = -\sum_0 \frac{\alpha^2}{E+\hbar\omega_0}$$

$$= -\frac{M_0 V}{4\pi c^3} \int_{\omega_0-\Delta\omega_0}^{\omega_0+\Delta\omega_0} \omega^2 d(\hbar\omega) \simeq \frac{M_0 V}{12\pi c^3} \Delta\omega_0[6\omega_0^2 + (\Delta\omega_0)^2]$$

This red shift is quite small. The remaining component of the second term of (27) is again complex,

$$\sum_0 \frac{\alpha^2}{E-\hbar\omega_0} = \Delta\epsilon_0 + i\gamma_0$$

where

$$\Delta\epsilon_0 = P\int \frac{\alpha^2}{(E-\hbar\omega_0)} \frac{\omega^2 d\omega}{2\hbar\pi^2 c^3}$$

and

$$2\gamma_0/\hbar = (2\pi/h)(\alpha^2\omega^2)/(2\hbar\pi^2 c^3)$$

the width at resonance. This width in MnF_2 is estimated to be about 14 G at K-band for a sample volume of about $1{\cdot}7\ mm^3$. The last term of the self energy which gives the interference can be evaluated only approximately near the resonance. Taking $E = \epsilon_0$ and $\Delta\epsilon'_0 = \Delta\epsilon_0 + \Delta c$

$$\Sigma_3 = \frac{\alpha^4 \left(\frac{1}{E+\hbar\omega_0} - \frac{1}{E-\hbar\omega_0}\right)^2}{E+\epsilon_0 - \sum_k' \frac{F_k^2}{E+\epsilon_k} - \sum_0 \alpha^2 \left(\frac{1}{E+\hbar\omega_0} - \frac{1}{E-\hbar\omega_0}\right)}$$

$$= (\Delta\epsilon'_0 + i\gamma_0)^2 [2\epsilon_0 - \Delta\epsilon_k^+ - (\Delta\epsilon_0' + i\gamma_0)]^{-1}$$

where

$$\Delta\epsilon_k^+ = \sum_k' \frac{F_k^2}{E+\epsilon_k} = \int \frac{F_k^2 \rho_k dk}{2\epsilon_k}$$

is real We further set, $\Delta c'_0 + \Delta\epsilon_k^+ = \Delta\epsilon_0''$; $2\epsilon_0 - \Delta\epsilon_0'' = 2\epsilon_0''$; so that

$$\Sigma_3 = (\Delta\epsilon_0'^2 - \gamma_0^2 + 2i\gamma_0\Delta\epsilon_0')(2\epsilon_0'' - i\gamma_0)^{-1}.$$

The real and imaginary parts of the above are

$$\mathrm{Re}\,\Sigma_3 = \{2\epsilon_0''(\Delta\epsilon_0'^2 - \gamma_0^2) - 2\gamma_0^2\Delta\epsilon_0'\}[(2\epsilon_0'')^2 + \gamma_0^2]^{-1}$$

$$\mathrm{Im}\,\Sigma_3 = \{\gamma_0(\Delta\epsilon_0'^2 - \gamma_0^2) + 2\gamma_0\Delta\epsilon_0'(2\epsilon_0'')\}[(2\epsilon_0'')^2 + \gamma_0^2]^{-1}$$

for $\Delta\epsilon_0'$ γ_0 14G at X-band, the contribution to the line width for a well polished sample when $\Delta\epsilon_k^+$ is negligible, is less than 1G. However, if the sample is rough, $\Delta\epsilon_k^+$ can be large and accordingly the contribution to the line width from Σ_3 can rise upto about 4G or so. From this calculation our conclusion is that though interference between the magnon-photon interaction and the magnon surface scattering does occur, its effect on the magnon-photon interaction is small and the two processes can be treated independently. Upto the second-order there is no interference at all. The interference which we obtained is in the sixth order. The dispersion relation $E - \epsilon_0 - \Sigma = 0$, contains the analogues of the polaritons but the effect of the surface scattering can be ignored atleast in the problem that we have worked out. This is because our materials are transparent and microwaves are well penetrated so that the bulk of the magnons are in the interior of the sample. This is not the case with metals where surface effects present a problem by themselves.

5. Magnetic superradiance

We define the Dicke operators (Dicke 1954),

$$R^+ = b^\dagger (N - b^\dagger b)^{1/2}$$
$$R^- = (N - b^\dagger b)^{1/2} b$$
$$R_3 = b^\dagger b - N/2. \tag{31}$$

If we compare this with the site variables it appears that $2S$ is replaced by N. So we are dealing with $S = N/2$ system. The probability of simultaneous detection of m photons is proportional to $< R^{+m}R^{-m}(t) >$ (Ressayre and Tallet 1975, Banfi and Bonifacio 1975, Bonifacio and Lugiato 1975). Since the expectation values are linear superpositions for the moments $< b^{+m}b^{-m}(t) >$ leads to the knowledge of the statistical behaviour of the radiation field. In the limit of large N the value $< R^{+m}R^{-m}(t) >$ behaves classically only if $< b^{+k}b^{-k}(t) >$ does, $(1 \leqslant m \leqslant 2k)$. In the case of Dicke superradiance it has been found (Ressayre and Tallet 1975), that

$$< b^{\dagger}b(t) > = \tfrac{1}{2}N[1-\tanh \tfrac{1}{2}(t-t_0)] \tag{32}$$

where

$$t_0 = (N\gamma)^{-1} \ln [n_0/(N-n_0)]. \tag{32a}$$

The density operator $\rho(t)$ is introduced through Sudarshan-Glauber coherent states $\rho(0) = | n_0 > < n_0 |$ with $n_0 \ll N$. Using Agarwal-Wolf relations (Agarwal and Wolf 1970), the superradiant intensity is found to be,

$$< R^{+k}R^{-k}(t) > = [\tfrac{1}{2}N \operatorname{sech} \tfrac{1}{2}N_{\gamma}(t-t_0)]^{2k} \tag{33}$$

which corresponds to the emission of a superradiant pulse proportional to $N^2/4$ at time t_0, taking single particle $k = 1$. In the case of magnons we introduce the Fourier transforms of the site variables and then put the Bogoliubov transformation. The single atom variables lead to

$$< a_i^{\dagger}a_j > = N^{-1} < \sum_k a_k^{\dagger}a_k > = < \alpha_k^{\dagger}\alpha_k > \tag{34}$$

For classical behaviour, $< b^{+m}b^{-m} > = < b^{+}b >^{m}$. In our case,

$$< \alpha_k^{+}\alpha_k > = n_k = [\exp(\hbar\omega/k_BT)-1]^{-1} \tag{35}$$

so the magnetic superradiant pulse is emitted with intensity

$$I_{ms} \propto n_k^2. \tag{36}$$

If the magnetic material is highly anisotropic, the magnon velocities are anisotropic. If the g-values have large anisotropy, all magnons will travel in a preferred direction along which the velocity is the largest. In one-dimensional magnetic systems of which there are many real materials known, magnons travel only in one direction because the inter-chain coupling is very weak. In the case of a Dicke superradiance from a cylindrical sample the radiation though proportional to N^2 is divergent in a cone of angle λ/d, where d is the diameter of the sample holder. A divergent beam is emitted with cross section $N^2\{\pi[L \tan(\lambda/2d) + d/2]^2\}^{-1}$ at a distance L from the terminating end of the sample. As L increases the cross section decreases as the inverse square of this distance, if $d \ll L$. On the other hand magnons are ordered so instead of a divergent beam with cone angle λ/d, we have a strong beam of light if the magnetic material is cut such that the magnon velocity is largest along the axis of the cylindrically shaped sample. The cross section of the magnetic superradiance is $n_k^2[\pi(d/2)^2]^{-1}$, independent of L. The ratio,

$$I_{ms}/I_{Dicke} = 4n_k^2[L\tan(\lambda/2d)+d/2]^2/(N^2d^2). \tag{37}$$

At large distance $d \ll L \tan(\lambda/2d)$ and since the number of magnons is of the same order as the number of atoms $n_k \simeq N$,

$$I_{ms}/I_{Dicke} = 4L^2\tan^2(\lambda/2d)/d^2 \tag{38}$$

which can be a very large number. At $L = 10d$, $I_{ms}/I_{Dicke} \simeq 4\times10^2\tan^2(\lambda/2d)$ If $\lambda = 1$ cm, $d = 1$ cm, $I_{ms}/I_{Dicke} \simeq 10^2$. If $d = 0{\cdot}1$ cm, $\lambda = 1$ cm, $\tan^2(\lambda/2d)$ $= 11{\cdot}4$ and $I_{ms}/I_{Dicke} \simeq 10^3$. If $d = \lambda/\pi$, $I_{ms}/I_{Dicke} \rightarrow \infty$, so we can identify our magnetic superradiance as hyperradiance in which the photon density is much more than in the Dicke superradiance As n_k is a function of temperature, the hyperradiant intensity depends on temperature whereas the superradiance is independent of temperature. As the temperature is varied there is a phase transition from the ordered state to the disordered paramagnetic state. Therefore there will be a phase transition from the hyperradiance to the superradiance. In the normal configuration the resonance occurs at $\omega = c_0$ whereas in the parallel pumping the resonance occurs at $\omega = c + \epsilon_-$, i e. at $\omega/2$. So if the orientation of the incident photon beam is changed our hyperradiance can be tuned from ω to $\omega/2$ where ω is at the gap in the magnon dispersion. Most of the modern lasers and masers use disordered systems such as ruby or some gas, say CO_2 and the resulting laser is divergent. Our idea essentially is that if the system was ordered or preferencially polarized the pulse divergence could be considerably minimized.

6. Higher order effects

Suhl has discovered two effects. The first effect makes use of one-magnon one-photon particle number conserving terms and the dipole-dipole magnon splitting term,

$$\mathcal{H}'_d = \sum_{k_1k_2k_3} f_{k_1k_2k_3}\, a_{k_1} a^\dagger_{k_2} a^\dagger_{k_3} \delta(k_1-k_2-k_3) + h.c \tag{39}$$

This occurs in the fourth-order perturbation theory as shown in Figure 2 marked as Suhl (1). The second effect of Suhl makes use of one-magnon one-photon number conserving term and the exchange interaction (Oguchi 1960),

$$\mathcal{H}_e' = \sum_{k_1k_2k_3k_4} [(Jz(SN)(\gamma_{k_1}+\gamma_{k_4}-2\gamma_{k_2-k_4})+(Jz/16NS)(\gamma_{k_1}+\gamma_{k_4})]$$

$$\times V(k_1k_2;\, k_3k_4)a^{\dagger}_{k_1}a^{\dagger}_{k_2}a_{k_3}a_{k_4}\delta(k_1+k_2-k_3-k_4) \tag{40}$$

and occurs in the sixth-order perturbation theory marked as Suhl (2) in Figure 2. It goes without saying that our effects discussed in the previous section occur in the second order. If we go to higher orders there are several effects which occur along with the Suhl (1).

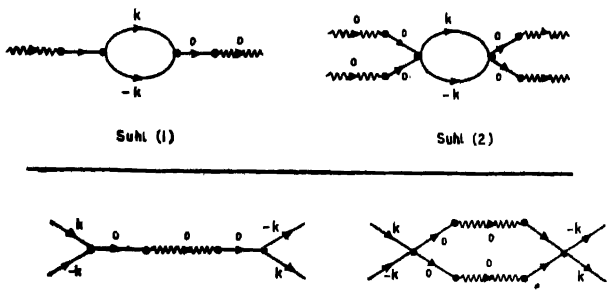

Figure 2. Suhl's effects are shown to contribute only in the fourth and sixth order. The upper figures were indicted by Keffer. These are not gauge in variant. Therefore the lower figures must be added. The sums are invariant.

7. Conclusions

We have constructed the magnon-photon interaction for the magnetic vector of the radiation field perpendicular with the direction of magnetization in a ferromagnetic insulator and from that we calculated the one-magnon self energy. We indicated the frequency shift and life times of magnons due to new second-order radiation effects. It is found that in insulators the interference due to surface scattering is small. The possibility of hyperradiance from magnetically ordered materials is pointed out. Brief remarks are made on the existence of higher-order effects and it is noted that there are several new effects which compete with the effects of Suhl which occur in the fourth and sixth order. It is clear that there are many events to occur before Suhl's effects come into play. The reason that Suhl did not see the lower-order effects and many higher-order effects is perhaps that his treatments were semiclassical in which the magnetic moment was treated as a vector and commutation effects were ignored. We have constructed the magnon-photon interaction for the first time and have looked into the effects from the proper quantum mechanical viewpoint. We are thus able to discover several new radiation processes.

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